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NUMERICAL RESULTS FOR ADVECTION-DOMINATED HEAT TRANSFER IN A MOVING FLUID WITH A NON-SLIP BOUNDARY CONDITION

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ABSTRACT

This paper is concerned with the laminar transfer of heat by forced convection where the velocity profile is taken to be parabolic. In the advection dominated case the problem is described mathematically by a singularly perturbed boundary value problem with a non-slip condition. It has been established both theoretically and computationally that numerical methods composed of upwind finite difference operators on special piecewise uniform meshes have the property that they behave uniformly well, regardless of the magnitude of the ratio of the advection term to the diffusion term. A variety of choices of special piecewise uniform mesh is examined and it is shown computationally that these lead to numerical methods also sharing this property. These results validate a previous theoretical result which is quoted.

KEY WORDS Laminar heat transfer Non-slip boundary condition

INTRODUCTION

The transfer of heat by the stationary flow of an incompressible fluid in an n-dimensional domain Ω ($n = 2, 3$) gives rise to a temperature distribution which satisfies the equation:

$$-\kappa\Delta T(x) + \rho_0 C_p \bar{u}(x)\cdot\nabla T(x) = Q(x) \qquad x\in\Omega \tag{1a}$$

where Δ is the Laplacian operator, κ is the thermal conductivity of the fluid, ρ_0 and C_p are the density and specific heat (at constant pressure), $Q(x)$, $\bar{u}(x)$, $T(x)$ are respectively the density of heat sources, fluid velocity, and temperature at the point x. On the solid surface $\partial\Omega$ bounding the fluid, the temperature of the fluid is given by:

$$T(x) = \phi(x) \qquad x\in\partial\Omega \tag{1b}$$

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This problem can be transformed into the non-dimensional form:

$$-\varepsilon\Delta v(\dot{x}) + \bar{U}(\dot{x})\cdot\nabla v(\dot{x}) = g(\dot{x}) \qquad \dot{x}\in\hat{\Omega} \tag{2a}$$

$$v(\dot{x}) = \dot{\phi}(\dot{x}) \qquad \dot{x}\in\partial\hat{\Omega} \tag{2b}$$

where $\hat{\Omega}$ is the transformed domain with boundary $\partial\hat{\Omega}$, $\dot{x} = \dot{x}(x)$, $x_s = x_s/l$, $s = 1, \ldots, n$ is the new non-dimensional variable, and

$$\bar{U}(\dot{x}) = \bar{u}(\dot{x})/u_0 \tag{3}$$

where l, u_0 are respectively typical sizes of the domain Ω and the fluid velocity $\bar{u}$. The parameter ε and the function $v(\dot{x})$ are given by:

$$\varepsilon = \frac{\kappa}{C_p\rho_0 u_0 l}, \qquad v(\dot{x}) = T(\dot{x})/T_0$$

and T_0 is a typical value of the temperature. The parameter $\varepsilon = 1/Pe$ where Pe is the Peclet number. Note that (2a) is non-linear if:

$$\bar{U}(x) = \bar{v}(\dot{x}) \qquad \dot{x}\in\hat{\Omega}$$

The parameter ε in (2a) may be arbitrarily small. In cases where ε tends to zero, boundary layers may appear in some neighbourhoods of the physical boundary of the fluid. This is due to the fact that in such cases (2a) is a singularly perturbed differential equation. If the boundary of the fluid is fixed, then:

$$\bar{U}(\dot{x}) = 0 \qquad \dot{x}\in\partial\hat{\Omega} \tag{4}$$

In a number of applied problems, e.g. a fluid moving between surfaces or in tubes, the velocity vector has a parabolic profile in the direction orthogonal to the streamlines. The boundary layers which appear in such problems are described by equations of parabolic type of dimension $n - 1$, where the variable along the streamlines plays the role of the time variable, and hence they are called parabolic layers.

Two distinct classes of parabolic layers may be identified. The first is the class of non-degenerate parabolic boundary layers where $|\bar{u}(x)| > 0$ in a closed neighbourhood of the relevant part of the boundary. Layers in this class are described by singularly perturbed parabolic equations with coefficients which are strictly greater than zero. The second is the class of degenerate parabolic boundary layers where $|\bar{u}(x)| > 0$ in an open neighbourhood of the relevant part of the boundary and $\bar{u}(x) = 0$ on that part of the boundary. Layers in this class are described by singularly-perturbed parabolic equations in which the coefficients multiplying the time-like derivatives are equal to zero on the physical boundary. This means that the parabolic equations degenerate on this part of the boundary to equations of elliptic type. It follows that if the velocity $\bar{u}$ satisfies the no-slip condition (4) on the physical boundary, the corresponding parabolic layer is degenerate. For a discussion of various kinds of boundary layers and their asymptotic behaviour, see Vishik and Lyusternik[5], Il'in[2], Shih and Kellogg[3] and the references cited therein.

In this paper, problems of the form (2) in two dimensions are examined and an ε-uniform finite difference method is described and investigated. The accuracy of the method does not depend on the value of the parameter ε, but only on the number of nodes used. The maximum error tends to zero as the number of nodes increases. This means that the accuracy depends only on the amount of computation performed. Theoretical results for both non-degenerate and degenerate parabolic boundary layers are presented in Shishkin[4], where ε-uniform finite difference methods are also constructed. A preliminary version of this work appeared in Hegarty *et al.*[1].

PROBLEM FORMULATION

In the unit square

$$\Omega = (0, 1) \times (0, 1) \tag{5}$$

with the boundary $\partial\Omega = \bar{\Omega}\backslash\Omega$, consider the boundary value problem

$$\left(\varepsilon\Delta - x_2(1 - x_2)\frac{\partial}{\partial x_1}\right)u(x) = f(x) \qquad x \in \Omega \tag{6a}$$

$$u(x) = \phi(x) \qquad x \in \partial\Omega_D \tag{6b}$$

$$\frac{\partial}{\partial n}u(x) = \psi(x) \qquad x \in \partial\Omega_O \tag{6c}$$

where

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_O, \qquad \partial\Omega_O = \{x : x_1 = 1, 0 < x_2 < 1\}$$

and $\partial/\partial n$ is the derivative normal to $\partial\Omega_O$, i.e., $\partial/\partial n = \partial/\partial x_1$. The functions f, ϕ and ψ are assumed to be sufficiently smooth. The parameter ε takes any value in (0, 1]. Compatibility conditions on the boundary data sufficient to guarantee a smooth solution are assumed at the corners of $\partial\Omega$. It is also assumed that the function $f(x)$ satisfies:

$$|f(x)| \leqslant Mx_2(1 - x_2) \qquad x \in \bar{\Omega}$$

Here and below, M, m denote sufficiently large, respectively small, generic positive constants which are independent of the parameter ε. In the case of the discrete problem, these constants are also independent of the mesh parameters. This condition ensures that the solution of (6) is bounded ε-uniformly.

The two edges $\partial\Omega_C = \{x : x_2 = 0,1; 0 < x_1 < 1\}$ are characteristics of the reduced equation (6a) with $\varepsilon = 0$. As the parameter ε tends to zero, boundary layers appear in the neighbourhood of this part of the boundary. These are described by the parabolic equation:

$$\varepsilon\frac{\partial^2}{\partial x_2^2}u(x) - x_2(1 - x_2)\frac{\partial}{\partial x_1}u(x) = 0 \qquad x \in \Omega$$

The coefficient of $\partial u/\partial x_1$ is equal to zero on the set $\partial\Omega_C$. This part of the boundary is parallel to the streamlines.

In the next section a finite difference method is constructed for problem (6) whose solutions converge ε-uniformly on a special piecewise-uniform mesh condensing in the neighbourhood of the boundary layer. The recipe for condensing the nodes depends on *a priori* estimates of the solution and its derivatives given in Hegarty *et al.*[1].

In the neighbourhood of the boundary layer, derivatives in a direction normal to the boundary $\partial\Omega_C$ (the characteristic part of the boundary) increase unboundedly as ε tends to zero. In a neighbourhood of the boundary layer, the natural variables are $\rho_2(x)\varepsilon^{-1/3}$, x_1 where $\rho_2(x)$ is the distance from the point $x \in \Omega$ to the set $\partial\Omega_C$. Outside a neighbourhood of the boundary layer, the derivatives in the direction normal to the boundary $\partial\Omega_C$ are bounded ε-uniformly. The partial derivatives for the regular part of the solution and for the first term in the expansion of the singular part of the solution (i.e. the partial derivatives in the direction tangential to the boundary $\partial\Omega_C$) are bounded ε-uniformly.

SPECIAL FINITE DIFFERENCE METHODS

To solve the boundary value problem (6) we use standard finite difference operators on special piecewise-uniform meshes. On the set $\bar{\Omega} = \Omega \cup \partial\Omega$ we introduce the special mesh

$$\bar{\Omega}_N^* = \omega_1 \times \bar{\omega}_2^* \tag{7}$$

where $\bar{\omega}_1$ is a uniform mesh

$$\bar{\omega}_1 = \{x_1^i = i/N_1\}_{i=0}^{N_1}, \qquad x_1^0 = 0, \qquad x_1^{N_1} = 1$$

and $\bar{\omega}_2^*$ is a special piecewise uniform mesh constructed as follows. The interval [0, 1] is divided into three parts:

$$[0, \sigma], \qquad [\sigma, 1-\sigma], \qquad [1-\sigma, 1]$$

where $\sigma \in (0, 1/4]$ depends on ε and N_2 and is given by:

$$\sigma = \min[1/4, m\varepsilon^{1/3}g(N_2)] \tag{8}$$

where m is an arbitrary positive number and $g(N_2)$ is an arbitrary increasing function $g(N_2) \to \infty$ and $g(N_2)N_2^{-1} \to 0$ as $N_2 \to \infty$. The intervals $[0, \sigma]$, $[1-\sigma, 1]$ are divided into $N_2/4$ equal parts and the interval $[\sigma, 1-\sigma]$ is divided into $N_2/2$ equal parts.

To solve problem (6) we use the finite difference method:

$$\begin{cases} \varepsilon(\delta_1^2 + \delta_2^2)z(x) - x_2(1-x_2)D_1^- z(x) = f(x) & x \in \Omega_N^* \\ z(x) = \phi(x) & x \in \partial\Omega_{D,N}^* \\ D_1^- z(x) = \psi(x) & x \in \partial\Omega_{O,N}^* \end{cases} \tag{9}$$

to find a numerical approximation $z(x)$ to the solution $u(x)$. Here $\delta_s^2 z(x)$, $s = 1, 2$ are second-order differences defined by:

$$\delta_s^2 z(x) = (D_s^+ z(x) - D_s^- z(x))/h_s$$

and $D_s^+ z(x)$, $D_s^- z(x)$ are forward, backward differences.

The ε-uniform convergence and error estimate for this numerical method are contained in the following theorem.

Theorem 1. *The solutions of the finite difference method consisting of the classical finite difference operator (9) on the special piecewise uniform mesh (7) converge ε-uniformly as $N_2 \to \infty$ to the solution of problem (6) provided that $g(N_2) = \sqrt{N_2}$. Moreover, the following error estimate holds with this choice of the function g:*

$$|u(x) - z(x)| \leqslant MN^{-1/6} \qquad x \in \bar{\Omega}_N^*$$

A proof of this theorem may be found in Shishkin[4]. It should be noted that the solutions of the finite difference method (9), (7) converge ε-uniformly for any increasing function $g(N)$, $g(N) \to \infty$ and $g(N)N^{-1} \to 0$ as $N \to \infty$. In the next section, this will be verified numerically, by examining the convergence properties of the numerical solutions, from various choices of the function $g(N)$, applied to two particular examples of problem (6).

COMPUTATION OF THE TEMPERATURE DISTRIBUTION IN A MOVING LIQUID: AN EXAMPLE

Suppose it is required to find the temperature distribution in a liquid flowing between two parallel plates a distance l apart. Consider a section of the flow of width l orthogonal to the direction of the flow. The temperature distribution is described by (1a). After a transformation, the equations may be written in the form (2).

Let the distance l between the plates be 10^{-2} m, the density of the liquid $\rho_0 = 10^3$ kg m^{-3}, the thermal conductivity $\kappa = 0.58$ W (mK)$^{-1}$ and the specific heat $C_p = 4190$ J (kg K)$^{-1}$. It follows that $\varepsilon = 1.385 \times 10^{-5} u_o^{-1}$ m s^{-1}, and so if $u_0 = 1.385 \times 10^{-2}$ ms^{-1}, then $\varepsilon = 10^{-3}$. Let the maximum temperature on the surface be $T_E = 100$°C, and let the density of the heat sources in the medium be zero ($Q(x) \equiv 0$). Suppose that the temperature on the upper edge changes from 0 to 100°C, according to: $T(x) = (T_E x_1)/l, 0 \leqslant x_1 \leqslant l, x_2 = 1$, and that the temperature is zero both on the lower edge and at the inflow boundary. A natural (Neumann) boundary condition is given at the outflow boundary: $\partial T/\partial x_1 = 0, x_1 = l$. Let the symbol x now represent the undimensionalized coordinate. The above leads to the following boundary value problem for the non-dimensional variable $u(x) = T(x)/T_0$, where $T_0 = T_E = 100$°C:

$$\begin{aligned} &\left(\varepsilon\Delta - x_2(1-x_2)\frac{\partial}{\partial x_1}\right)u(x) = 0, x \in \Omega = (0,1) \times (0,1) \\ &u(x) = x_1, \text{for } x_2 = 1, \text{and } u(x) = 0, x_1 = 0 \text{ or } x_2 = 0 \\ &\frac{\partial}{\partial x_1}u(x) = 0 \qquad x_1 = 1 \end{aligned} \tag{10}$$

The special mesh $\bar{\Omega}_N^*$ consists of a piecewise uniform mesh in the x_2-direction and a uniform mesh in the x_1-direction. In the x_2-direction, the interval [0, 1] is subdivided into three subintervals:

$$[0, \sigma], [\sigma, 1-\sigma], [1-\sigma, 1] \text{ where } \sigma \equiv \min\{\varepsilon^{1/3}g(N), \tfrac{1}{4}\}$$

On each of the three subintervals a uniform mesh is used with: $N/2$ mesh points in $[\sigma, 1-\sigma]$, $N/4$ mesh points in $[0, \sigma]$, and $[1-\sigma, 1]$. We consider the three cases $g(N) \equiv \ln N$, $\sqrt{N}$ and $N^{1/3}$. With this distribution of the mesh points, the mesh $\bar{\Omega}_N^*$ tends to a uniform mesh when ε is large. The mesh $\bar{\Omega}_N^*$ is depicted in *Figure 1* in the case where $g(N) \equiv \ln N$, $N = 32$ and $\varepsilon = 10^{-5}$.

The following finite difference method with the above special mesh $\bar{\Omega}_N^*$ will now be considered:

$$\begin{cases} \varepsilon(\delta_1^2 + \delta_2^2)z(x) - x_2(1-x_2)D_1^- z(x) = 0, x \in \Omega_N^* \\ z(x) = x_1, \text{for } x_2 = 0, \text{and } z(x) = 0, x_1 = 0 \text{ or } x_1 = 1 \\ D_1^- z(x) = 0 \qquad x \in \partial\Omega_{O,N}^* \end{cases} \tag{11}$$

The solution of (11) with $\varepsilon = 10^{-5}$, $N = 32$ and $g(N) \equiv \ln N$ is shown in *Figure 2*. It can be observed that the solution is zero except in the neighbourhood of the boundary layer at the edge $\{x : x_2 = 1; 0 < x_1 < 1\}$. Equation (11) is solved on a sequence of meshes, with $N = 8$, 16, 32, 64, 128, 256, where N is the number of mesh elements used in both directions. The errors

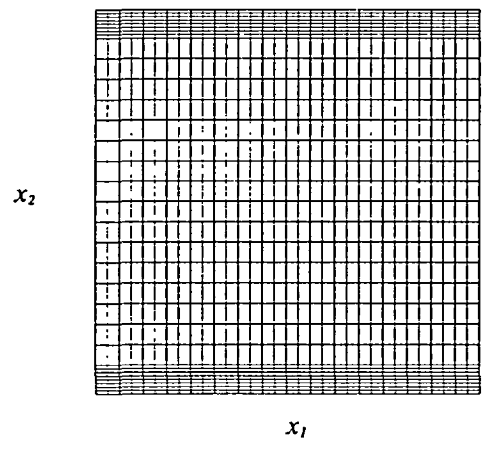

Figure 1 Piecewise uniform mesh Ω_N^* with $N = 32$, $\varepsilon = 10^{-5}$ and $g(N) = \ln N$

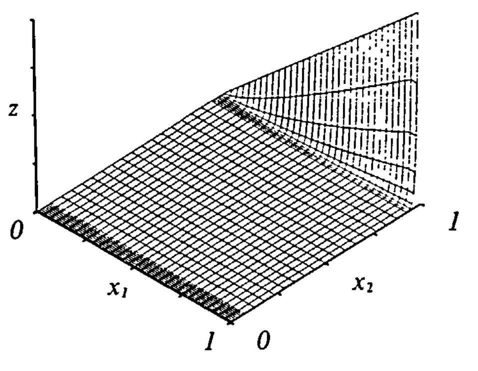

Figure 2 Numerical solution of (11) with $N = 32$, $\varepsilon = 10^{-5}$ and $g(N) = \ln N$

$|u(x_i, y_j) - z(x_i, y_j)|$ are approximated for successive values of ε on the five coarser meshes by $\varepsilon_\varepsilon^N(i,j) = |z_I^{256}(x_i, y_j) - z^N(x_i, y_j)|$, where the superscript indicates the number of mesh elements used and the subscript I denotes interpolation. For each ε and N the maximum nodal error is approximated by:

$$E_{\varepsilon,N} = \max_{i,j} \varepsilon_\varepsilon^N(i,j)$$

For each N define:

$$E_N = \max E_{\varepsilon,N}$$

Table 1 $E_{\varepsilon,N}$ and E_N for (11) on the special mesh Ω_N^* with $g(N) = \ln N$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.627×10^{-1}	0.322×10^{-1}	0.156×10^{-1}	0.688×10^{-2}	0.240×10^{-2}
2^{-2}	0.644×10^{-1}	0.327×10^{-1}	0.157×10^{-1}	0.691×10^{-2}	0.240×10^{-2}
2^{-4}	0.683×10^{-1}	0.339×10^{-1}	0.160×10^{-1}	0.698×10^{-2}	0.242×10^{-2}
2^{-4}	0.702×10^{-1}	0.352×10^{-1}	0.164×10^{-1}	0.707×10^{-2}	0.243×10^{-2}
2^{-8}	0.695×10^{-1}	0.377×10^{-1}	0.173×10^{-1}	0.729×10^{-2}	0.247×10^{-2}
2^{-10}	0.633×10^{-1}	0.417×10^{-1}	0.191×10^{-1}	0.780×10^{-2}	0.256×10^{-2}
2^{-12}	0.660×10^{-1}	0.438×10^{-1}	0.217×10^{-1}	0.883×10^{-2}	0.276×10^{-2}
2^{-14}	0.671×10^{-1}	0.444×10^{-1}	0.227×10^{-1}	0.961×10^{-2}	0.330×10^{-2}
2^{-16}	0.677×10^{-1}	0.446×10^{-1}	0.232×10^{-1}	0.101×10^{-1}	0.348×10^{-2}
2^{-18}	0.681×10^{-1}	0.447×10^{-1}	0.234×10^{-1}	0.104×10^{-1}	0.362×10^{-2}
2^{-20}	0.684×10^{-1}	0.447×10^{-1}	0.235×10^{-1}	0.106×10^{-1}	0.370×10^{-2}
2^{-22}	0.685×10^{-1}	0.447×10^{-1}	0.235×10^{-1}	0.106×10^{-1}	0.374×10^{-2}
2^{-24}	0.686×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.106×10^{-1}	0.375×10^{-2}
2^{-26}	0.686×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.107×10^{-1}	0.376×10^{-2}
2^{-28}	0.686×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.107×10^{-1}	0.376×10^{-2}
2^{-30}	0.687×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.107×10^{-1}	0.376×10^{-2}
2^{-32}	0.687×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.107×10^{-1}	0.376×10^{-2}
E_N	0.687×10^{-1}	0.447×10^{-1}	0.236×10^{-1}	0.107×10^{-1}	0.376×10^{-2}

Table 2 $E_{\varepsilon,N}$ and E_N for (11) on the special mesh Ω_N^* with $g(N) = \sqrt{N}$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.627×10^{-1}	0.322×10^{-1}	0.156×10^{-1}	0.688×10^{-2}	0.240×10^{-2}
2^{-2}	0.644×10^{-1}	0.327×10^{-1}	0.157×10^{-1}	0.691×10^{-2}	0.240×10^{-2}
2^{-4}	0.683×10^{-1}	0.339×10^{-1}	0.160×10^{-1}	0.698×10^{-2}	0.242×10^{-2}
2^{-6}	0.702×10^{-1}	0.352×10^{-1}	0.164×10^{-1}	0.707×10^{-2}	0.243×10^{-2}
2^{-8}	0.695×10^{-1}	0.377×10^{-1}	0.173×10^{-1}	0.729×10^{-2}	0.247×10^{-2}
2^{-10}	0.464×10^{-1}	0.417×10^{-1}	0.191×10^{-1}	0.780×10^{-2}	0.256×10^{-2}
2^{-12}	0.330×10^{-1}	0.427×10^{-1}	0.226×10^{-1}	0.883×10^{-2}	0.276×10^{-2}
2^{-14}	0.327×10^{-1}	0.431×10^{-1}	0.274×10^{-1}	0.110×10^{-1}	0.326×10^{-2}
2^{-16}	0.320×10^{-1}	0.440×10^{-1}	0.283×10^{-1}	0.135×10^{-1}	0.417×10^{-2}
2^{-18}	0.341×10^{-1}	0.418×10^{-1}	0.280×10^{-1}	0.121×10^{-1}	0.363×10^{-2}
2^{-20}	0.340×10^{-1}	0.418×10^{-1}	0.280×10^{-1}	0.122×10^{-1}	0.640×10^{-2}
2^{-22}	0.339×10^{-1}	0.418×10^{-1}	0.280×10^{-1}	0.122×10^{-1}	0.642×10^{-2}
2^{-24}	0.339×10^{-1}	0.418×10^{-1}	0.281×10^{-1}	0.122×10^{-1}	0.643×10^{-2}
2^{-26}	0.339×10^{-1}	0.418×10^{-1}	0.281×10^{-1}	0.122×10^{-1}	0.643×10^{-2}
2^{-28}	0.339×10^{-1}	0.418×10^{-1}	0.281×10^{-1}	0.122×10^{-1}	0.643×10^{-2}
2^{-30}	0.338×10^{-1}	0.418×10^{-1}	0.281×10^{-1}	0.122×10^{-1}	0.643×10^{-2}
2^{-32}	0.338×10^{-1}	0.418×10^{-1}	0.281×10^{-1}	0.122×10^{-1}	0.643×10^{-2}
E_N	0.702×10^{-1}	0.440×10^{-1}	0.283×10^{-1}	0.135×10^{-1}	0.643×10^{-2}

From *Tables 1, 2* and *3*, the solutions of the finite difference method (11) for each choice of $g(N)$ are ε-uniformly convergent. That is, for all values of ε, the maximum nodal error decreases as N increases, for N sufficiently large. For the particular problem (10), the last row of *Table 4* indicates numerically that the maximum error is guaranteed to be less than 1°C for $N \geqslant 64$ for all values of ε.

A numerical method is said to be ε-uniform of order p on the mesh Ω_N if

$$\max_{\Omega^N} |u(x) - z^N(x)| \leqslant CN^{-p}$$

where C and $p > 0$ are independent of ε and N. For each value of N and ε the order of convergence p is approximated numerically by:

$$p_{\varepsilon,N} = \log_2\left(\frac{D_{\varepsilon,N}}{D_{\varepsilon,2N}}\right) \quad \text{where } D_{\varepsilon,N} \equiv \max_{i,j} |z_I^{2N}(x_i, y_j) - z^N(x_i, y_j)|$$

For each value of N the quantities:

$$D_N = \max D_{\varepsilon,N}, \qquad p_N = \log_2\left(\frac{D_N}{D_{2N}}\right)$$

are computed and the value p_N is taken as an estimate of the ε-uniform convergence rate. Values of p_N and $\bar{p}$ for $g(N) = \ln N$, $\sqrt{N}$ and $N^{1/3}$ are presented in *Table 4*.

Table 3 $E_{\varepsilon,N}$ and E_N for (11) on the special mesh Ω_N^* with $g(N) = N^{1/3}$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.627×10^{-1}	0.322×10^{-1}	0.156×10^{-1}	0.688×10^{-2}	0.240×10^{-2}
2^{-2}	0.644×10^{-1}	0.327×10^{-1}	0.157×10^{-1}	0.691×10^{-2}	0.240×10^{-2}
2^{-4}	0.683×10^{-1}	0.339×10^{-1}	0.160×10^{-1}	0.698×10^{-2}	0.242×10^{-2}
2^{-6}	0.702×10^{-1}	0.325×10^{-1}	0.164×10^{-1}	0.707×10^{-2}	0.243×10^{-2}
2^{-8}	0.695×10^{-1}	0.377×10^{-1}	0.173×10^{-1}	0.729×10^{-2}	0.247×10^{-2}
2^{-10}	0.660×10^{-1}	0.417×10^{-1}	0.191×10^{-1}	0.780×10^{-2}	0.256×10^{-2}
2^{-12}	0.691×10^{-1}	0.436×10^{-1}	0.215×10^{-1}	0.883×10^{-2}	0.276×10^{-2}
2^{-14}	0.708×10^{-1}	0.448×10^{-1}	0.223×10^{-1}	0.986×10^{-2}	0.338×10^{-2}
2^{-16}	0.718×10^{-1}	0.451×10^{-1}	0.229×10^{-1}	0.104×10^{-1}	0.357×10^{-2}
2^{-18}	0.723×10^{-1}	0.453×10^{-1}	0.232×10^{-1}	0.106×10^{-1}	0.370×10^{-2}
2^{-20}	0.726×10^{-1}	0.453×10^{-1}	0.233×10^{-1}	0.108×10^{-1}	0.377×10^{-2}
2^{-22}	0.728×10^{-1}	0.453×10^{-1}	0.233×10^{-1}	0.108×10^{-1}	0.380×10^{-2}
2^{-24}	0.729×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.382×10^{-2}
2^{-26}	0.730×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.382×10^{-2}
2^{-28}	0.730×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.383×10^{-2}
2^{-30}	0.730×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.383×10^{-2}
2^{-32}	0.730×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.383×10^{-2}
E_N	0.730×10^{-1}	0.453×10^{-1}	0.234×10^{-1}	0.109×10^{-1}	0.383×10^{-2}

Table 4 Orders of convergence p_N

$g(N)$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$\ln N$	0.92	1.09	1.08	1.03
$\sqrt{N}$	0.08	0.81	1.05	1.12
$N^{1/3}$	0.97	1.04	1.04	1.02

From *Table 4* it can be seen that in the cases $g(N) = \ln N$ and $g(N) = N^{1/3}$, the finite difference operator (4.3) on the special mesh Ω_N^* is higher ε-uniformly convergent of first order. This is a considerably bigger rate than that predicted by Theorem 1. In the case $g(N) = \sqrt{N}$, N needs to be sufficiently large, that is greater than 32, for a ε-uniform convergence rate of 1 to be attained.

As a further test of the effectiveness of the ε-uniform method (9), numerical solutions of the following problem are obtained. The main difference between this and the first problem is a

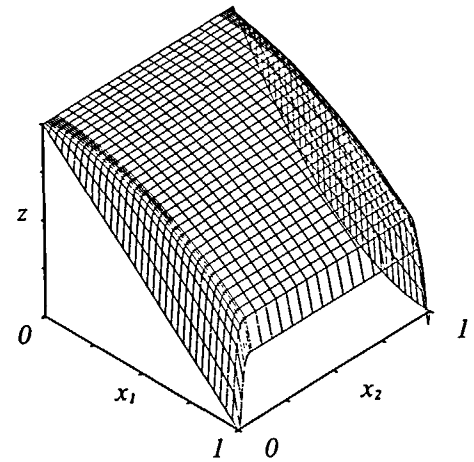

Figure 3 Numerical solution of (13) with $N = 32$, $\varepsilon = 10^{-5}$ and $g(N) = \ln N$

Table 5 $E_{\varepsilon,N}$ and E_N for (13) on the special mesh Ω_N^* with $g(N) = \ln N$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.701×10^{-1}	0.340×10^{-1}	0.160×10^{-1}	0.697×10^{-2}	0.241×10^{-2}
2^{-2}	0.748×10^{-1}	0.325×10^{-1}	0.163×10^{-1}	0.703×10^{-2}	0.242×10^{-2}
2^{-4}	0.102	0.495×10^{-1}	0.231×10^{-1}	0.987×10^{-2}	0.329×10^{-2}
2^{-6}	0.100	0.490×10^{-1}	0.230×10^{-1}	0.991×10^{-2}	0.331×10^{-2}
2^{-8}	0.102	0.482×10^{-1}	0.222×10^{-1}	0.950×10^{-2}	0.316×10^{-2}
2^{-10}	0.107	0.521×10^{-1}	0.235×10^{-1}	0.101×10^{-2}	0.333×10^{-2}
2^{-12}	0.111	0.552×10^{-1}	0.250×10^{-1}	0.107×10^{-1}	0.351×10^{-2}
2^{-14}	0.114	0.561×10^{-1}	0.256×10^{-1}	0.109×10^{-1}	0.366×10^{-2}
2^{-16}	0.115	0.565×10^{-1}	0.262×10^{-1}	0.111×10^{-1}	0.317×10^{-2}
2^{-18}	0.116	0.566×10^{-1}	0.265×10^{-1}	0.112×10^{-1}	0.376×10^{-2}
2^{-20}	0.116	0.567×10^{-1}	0.266×10^{-1}	0.114×10^{-1}	0.379×10^{-2}
2^{-22}	0.116	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.383×10^{-2}
2^{-24}	0.116	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.385×10^{-2}
2^{-26}	0.116	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.385×10^{-2}
2^{-28}	0.117	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
2^{-30}	0.117	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
2^{-32}	0.117	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
E_N	0.117	0.567×10^{-1}	0.267×10^{-1}	0.115×10^{-1}	0.386×10^{-2}

non-zero source term on the right-hand side of the transport equation (6).

$$\left(\varepsilon\Delta - x_x(1-x_2)\frac{\partial}{\partial x_1}\right)u(x) = x_1x_2(1-x_2) \qquad x\in\Omega = (0,1)\times(0,1)$$
$$u(x) = 1 - x_1, \text{for } x_2 = 0, u(x) = (1-x_1)^2 \qquad x_2 = 1 \text{ and } u(x) = 1, x_1 = 0 \qquad (12)$$
$$\frac{\partial}{\partial x_1}u(x) = -5, x_1 = 1$$

Table 6 $E_{\varepsilon,N}$ and E_N for (13) on the special mesh Ω_N^* with $g(N) = \sqrt{N}$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.701×10^{-1}	0.340×10^{-1}	0.160×10^{-1}	0.697×10^{-2}	0.241×10^{-2}
2^{-2}	0.748×10^{-1}	0.352×10^{-1}	0.163×10^{-1}	0.703×10^{-2}	0.242×10^{-2}
2^{-4}	0.102	0.495×10^{-1}	0.231×10^{-1}	0.987×10^{-2}	0.329×10^{-2}
2^{-6}	0.100	0.490×10^{-1}	0.230×10^{-1}	0.991×10^{-2}	0.331×10^{-2}
2^{-8}	0.102	0.482×10^{-1}	0.222×10^{-1}	0.950×10^{-2}	0.316×10^{-2}
2^{-10}	0.108	0.521×10^{-1}	0.235×10^{-1}	0.101×10^{-1}	0.333×10^{-2}
2^{-12}	0.103	0.594×10^{-1}	0.250×10^{-1}	0.107×10^{-1}	0.351×10^{-2}
2^{-14}	0.103	0.600×10^{-1}	0.286×10^{-1}	0.116×10^{-1}	0.370×10^{-2}
2^{-16}	0.103	0.603×10^{-1}	0.290×10^{-1}	0.126×10^{-1}	0.406×10^{-2}
2^{-18}	0.103	0.600×10^{-1}	0.288×10^{-1}	0.119×10^{-1}	0.501×10^{-2}
2^{-20}	0.103	0.601×10^{-1}	0.289×10^{-1}	0.119×10^{-1}	0.502×10^{-2}
2^{-22}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.503×10^{-2}
2^{-24}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.504×10^{-2}
2^{-26}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.504×10^{-2}
2^{-28}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.504×10^{-2}
2^{-30}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.504×10^{-2}
2^{-32}	0.103	0.602×10^{-1}	0.289×10^{-1}	0.120×10^{-1}	0.504×10^{-2}
E_N	0.108	0.603×10^{-1}	0.290×10^{-1}	0.126×10^{-1}	0.504×10^{-2}

Table 7 $E_{\varepsilon,N}$ and E_N for (13) on the special mesh Ω_N^* with $g(N) = N^{1/3}$

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1	0.701×10^{-1}	0.340×10^{-1}	0.160×10^{-1}	0.697×10^{-2}	0.241×10^{-2}
2^{-2}	0.748×10^{-1}	0.325×10^{-1}	0.163×10^{-1}	0.703×10^{-2}	0.242×10^{-2}
2^{-4}	0.102	0.495×10^{-1}	0.231×10^{-1}	0.987×10^{-2}	0.392×10^{-2}
2^{-6}	0.100	0.490×10^{-1}	0.230×10^{-1}	0.991×10^{-2}	0.331×10^{-2}
2^{-8}	0.102	0.482×10^{-1}	0.222×10^{-1}	0.950×10^{-2}	0.316×10^{-2}
2^{-10}	0.107	0.521×10^{-1}	0.235×10^{-1}	0.101×10^{-1}	0.333×10^{-2}
2^{-12}	0.111	0.545×10^{-1}	0.250×10^{-1}	0.107×10^{-1}	0.351×10^{-2}
2^{-14}	0.114	0.558×10^{-1}	0.253×10^{-1}	0.109×10^{-1}	0.372×10^{-2}
2^{-16}	0.115	0.563×10^{-1}	0.261×10^{-1}	0.111×10^{-1}	0.376×10^{-2}
2^{-18}	0.116	0.564×10^{-1}	0.264×10^{-1}	0.113×10^{-1}	0.377×10^{-2}
2^{-20}	0.116	0.565×10^{-1}	0.265×10^{-1}	0.114×10^{-1}	0.380×10^{-2}
2^{-22}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.384×10^{-2}
2^{-24}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.385×10^{-2}
2^{-26}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
2^{-28}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
2^{-30}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
2^{-32}	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.386×10^{-2}
E_N	0.117	0.565×10^{-1}	0.266×10^{-1}	0.115×10^{-1}	0.386×10^{-2}

Table 8 Orders of convergence p_N

$q(N)$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$\ln N$	1.14	1.08	1.02	1.00
$\sqrt{N}$	0.80	1.10	1.03	0.96
$N^{1/3}$	1.25	1.05	1.01	1.00

To solve this problem numerically, the following finite difference method with the special mesh $\bar{\Omega}_N^*$ will be considered:

$$\begin{cases} \varepsilon(\delta_1^2 + \delta_2^2)z(x) - x_2(1 - x_2)D_1^- z(x) = x_1 x_2(1 - x_2) & x \in \Omega_N^* \\ z(x) = 1 - x_1, \text{for}\, x_2 = 0, z(x) = (1 - x_1)^2 & x_2 = 1 \,\text{and}\, z(x) = 1, x_1 = 0 \\ D_1^- z(x) = -5, x \in \partial\Omega_{0,N}^* \end{cases} \tag{13}$$

The solution of (13) with $\varepsilon = 10^{-5}$, $N = 32$ and $g(N) = \ln N$ is shown in *Figure 3*. $E_{\varepsilon,N}$ and E_N are calculated as for (11) and shown in *Tables 5*, *6* and *7* for $g(N) = \ln N$, $\sqrt{N}$ and $N^{1/3}$, respectively.

From *Tables 5*, *6* and *7*, the solutions of the finite difference method (13) for each choice of $g(N)$ are ε-uniformly convergent. Again, for all values of ε, the maximum nodal error decreases as N increases.

From *Table 8* it can be seen that in the cases $g(N) = \ln N$ and $g(N) = N^{1/3}$, the finite difference operator (13) on the special mesh Ω_N^* is seen to be ε-uniformly convergent of first order. This is considerably higher rate than that predicted by Theorem 1. In the case $g(N) = \sqrt{N}$, N needs to be greater than 16, for a ε-uniform convergence rate of 1 to be attained.

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